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AN EXTENSION OF LEVERRIER-FADDEEV ALGORITHM USING A BASIS OF CLASSICAL ORTHOGONAL POLYNOMIALS

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To our friend Giuseppe Mastroianni with occasion of his 65^{th} *anniversary*

Abstract. In this paper, the Leverrier-Faddeev algorithm is extended in order to compute both the determinant $a(s)$ and the adjoint $B(s)$ of the polynomial matrix $s^pA_p + s^{p-1}A_{p-1} + \cdots + sA_1 + A_0$, where $A_i \in \mathbb{C}^{n\times n}$, $i = 1, 2, \ldots, p$. $B(s)$ and $a(s)$ are expressed in terms of a basis of classical orthogonal polynomials.

1. Introduction

When one considers a vibrating system consisting of N masses connected by linear springs of stiffness $(k_r)_{r=1}^N$ and the whole lies in a straight line on a smooth horizontal table and is excited by forces $(F_r(t))_{r=1}^N$, the Newton's equations of motion for the system are

$$\begin{aligned} m_r\ddot{u}_r &= F_r + \theta_{r+1} - \theta_r, \quad r = 1, 2, \ldots, N-1, \\ m_N\ddot{u}_N &= F_N - \theta_N. \end{aligned} \tag{1.1}$$

Hooke's law states that the spring forces are given by

$$\theta_r = k_r(u_r - u_{r-1}), \quad r = 1, 2, \ldots, N. \tag{1.2}$$

If the left hand end is pinned, then $u_0 = 0$.

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Forced vibration analysis concerns the solution of these equations for given forcing functions $F_r(t)$.

Free vibration analysis consists in finding solutions to the equations which require no external excitation, i.e., $F_r(t) \equiv 0$, $r = 1, 2, \ldots, N$, and which satisfy the stated end conditions (see [5]).

In order to express equations (1.1)–(1.2) in matrix form, we get

$$A\ddot{u} + Cu = F, \tag{1.3}$$

where $A = \operatorname{diag}(m_1, m_2, \ldots, m_N)$, $F = (F_1, F_2, \ldots, F_N)$,

$$C = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \cdots & 0 \\ -k_2 & k_2 + k_3 & -k_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & & & -k_N \\ 0 & \cdots & 0 & -k_N & k_N \end{bmatrix}$$

and $u = (u_1, u_2, \ldots, u_N)^T$. A and C are called, respectively, the inertia and stiffness matrices of the system.

If we assume initial conditions $u(0)$ and $u'(0)$ are given, then using the Laplace transform in (1.3) we get

$$(s^2 A + C)\tilde{u} = \tilde{F}(s) + sB + G,$$

where $\tilde{u}$ denotes the Laplace transform of the vector function u. Thus,

$$\tilde{u} = (s^2 A + C)^{-1}\tilde{F}(s) + (s^2 A + C)^{-1}(sB + G). \tag{1.4}$$

In order to solve the problem, we need to find the inverse of the polynomial matrix $s^2 A + C$ (see [1]). This question was analyzed in a more general context by several authors. In particular, in [4] an algorithm has been given to find the determinant of a general matrix polynomial as well as its adjoint matrix. The connection with the Leverrier's algorithm to find the determinant of $sI_N - A$ is shown. The aim of our contribution is twofold. First, we use an extension of the Leverrier's algorithm stated in [6], [7] for polynomial bases that is a more efficient computational method than the preceding one taking into account the use of the canonical basis in the linear space $\mathbb{P}$ of polynomials with complex coefficients. Second, we give an alternative approach to the results of [4], [10] assuming the leading coefficient of the polynomial matrix is any matrix, not necessary the unit matrix as there.

Notice that the inverse of the polynomial matrix in (1.4) is the transfer function of the linear system (1.1)–(1.2).

The structure of the paper is the following. In Section 2, we give the basic background about scalar orthogonal polynomials with a special emphasis in the so called classical orthogonal polynomials. Then, we describe the Leverrier's algorithm taking into account a polynomial basis constituted by classical orthogonal polynomials.

In Section 3 we introduce polynomial matrices as well as some special examples (orthogonal matrix polynomials), which are generated recursively. In Section 4 we give an algorithm to find the determinant of a polynomial matrix as well as the adjoint matrix. Furthermore, in Section 5, some examples are presented.

2. Classical Orthogonal Polynomials

Given a linear functional u in the linear space $\mathbb{P}$ of polynomials with complex coefficients, we call moments (u_n) the complex numbers $u_n = \langle u, x^n \rangle,\ n \in \mathbb{N}$. Here $\langle \cdot, \cdot \rangle$ denotes the duality bracket.

Definition 2.1. The linear functional u is said to be quasi-definite if the principal submatrices of the infinite Hankel matrix $H = (u_{j+k})_{j,k=0}^{+\infty}$ are nonsingular.

Thus, we can introduce an inner product associated with the quasi-definite linear functional u as follows:

$$\phi(p,q) = \langle u, pq \rangle, \quad p, q \in \mathbb{P}.$$

Proposition 2.1. ([2]) *The following statements are equivalent:*

(i) *u is a quasi-definite linear functional.*

(ii) *There exists a sequence of monic polynomials (P_n) such that*

– $\deg P_n = n$*,*

– $\phi(P_n, P_m) = 0$ *if* $n \neq m$,

– $\phi(P_n, P_n) \neq 0$ *for every* $n \in \mathbb{N}$.

(iii) *There exist sequences of complex numbers* (β_n) *and* γ_n *with* $\gamma_n \neq 0$ *for every* $n \in \mathbb{N}$ *such that the sequence of monic polynomials* (P_n) *defined by*

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n \geqslant 1 \tag{2.1}$$

satisfy $\langle u, P_n P_m \rangle = 0, n \neq m$, *and* $\langle u, {P_n}^2 \rangle \neq 0$, *for some linear functional* u.

The matrix representation of the operator associated with the multiplication by x in $\mathbb{P}$ is a tridiagonal matrix

$$J = \begin{bmatrix} \beta_0 & 1 & 0 & \cdots & \\ \gamma_1 & \beta_1 & 1 & \ddots & \\ 0 & \gamma_2 & \beta_2 & 1 & \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ & & & & \end{bmatrix}.$$

Notice that the characteristic polynomial of the principal submatrix J_n of dimension n is the polynomial P_n. The recurrence relation (2.1) reads as follows: the determinant of the matrix polynomial $sI_{n+1} - J_{n+1}$ can be given recursively as a linear combination of the determinants of the matrix polynomials $sI_n - J_n$ and $sI_{n-1} - J_{n-1}$. Examples of orthogonal polynomials with respect to a linear functional appear in Sturm-Liouville problems for second and fourth order linear differential equations with polynomial coefficients. More precisely, if u is a quasi-definite linear functional, then the following statements are equivalent

Proposition 2.2. ([8]) (i) *There exist polynomials* ϕ *and* ψ *with* $\deg \phi \leqslant 2$ *and* $\deg \psi = 1$ *such that the distributional equation*

$$D(\phi u) = \psi u, \tag{2.2}$$

holds.

(ii) *The sequence of monic polynomials* (P_n) *orthogonal with respect to* u *satisfies a second order linear differential equation*

$$\phi P_n'' + \psi P_n' + \lambda_n P_n = 0. \tag{2.3}$$

The equation (2.2) is called a Pearson differential equation. (2.3) means essentially that (P_n) is a hypergeometric family of polynomials. The linear functional u satisfying a Pearson equation is said to be classical. The sequence of orthogonal polynomials with respect to u is said to be a classical sequence of orthogonal polynomials.

For our purposes, we can use other two characterizations of classical orthogonal polynomials.

Proposition 2.3. ([8]) *The following statements are equivalent:*

(i) (P_n) *is a sequence of classical orthogonal polynomials.*

(ii) *The sequence of monic derivatives of* (P_n) *is also a sequence of orthogonal polynomials. We will denote* $Q_n = \dfrac{P'_{n+1}}{n+1}$.

(iii) *In the expansion of* P_n *in terms of the polynomial basis* (Q_n)*, the coefficients for* $0 \leqslant k \leqslant n-3$ *vanish. In other words,*

$$P_n(x) = Q_n(x) + r_n Q_{n-1}(x) + s_n Q_{n-2}(x). \tag{2.4}$$

The sequences of classical orthogonal polynomials, up to a linear change of variables, are

1. Hermite polynomials (H_n)

$$H_n(x) = \frac{H'_{n+1}(x)}{n+1}.$$

2. Laguerre polynomials (L_n^α), $\alpha > -1$,

$$L_n^\alpha(x) = \frac{\left(L_{n+1}^\alpha\right)'(x)}{n+1} + n\frac{\left(L_n^\alpha\right)'(x)}{n}.$$

3. Jacobi polynomials $\left(J_n^{(\alpha,\beta)}\right)$, $\alpha, \beta > -1$,

$$\begin{aligned} J_n^{(\alpha,\beta)}(x) &= \left(\frac{J_{n+1}^{(\alpha,\beta)}}{n+1}\right)'(x) + \frac{2n(\alpha-\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\left(\frac{J_n^{(\alpha,\beta)}}{n}\right)'(x) \\ &- \frac{4n(n-1)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}\left(\frac{J_{n-1}^{(\alpha,\beta)}}{n-1}\right)'(x). \end{aligned}$$

4. Bessel polynomials (B_n^α), $-\alpha \notin \mathbb{N}$,

$$\begin{aligned} B_n^\alpha(x) &= \left(\frac{B_{n+1}^\alpha}{n+1}\right)'(x) + \frac{4n}{(2n+\alpha)(2n+\alpha+2)}\left(\frac{B_n^\alpha}{n}\right)'(x) \\ &\quad + \frac{4n(n-1)}{(2n+\alpha-1)(2n+\alpha)^2(2n+\alpha+1)}\left(\frac{B_{n-1}^\alpha}{n-1}\right)'(x). \end{aligned}$$

Finally, we need an expression for $x^m P_n(x)$, $n, m \in \mathbb{N}$, in terms of (P_n). If we use the three-term recurrence formula (2.1) m-times, then it is straightforward to verify that

$$x^m P_n(x) = \sum_{k=n-m}^{n+m} \alpha_k^{n,m} P_k(x), \tag{2.5}$$

where $\alpha_k^{n,m} = \langle u, f_m P_n P_k\rangle / \langle u, P_k^2\rangle$ and $f_m(x) = x^m$. Notice that $\alpha_{n+m}^{n,m} = 1$ and

$$\begin{aligned} \alpha_k^{n,m} &= \frac{\langle u, f_{m-1} P_n f_1 P_k\rangle}{\langle u, P_k^2\rangle} \\ &= \frac{\langle u, f_{m-1} P_n \left(P_{k+1} + \beta_k P_k + \gamma_k P_{k-1}\right)\rangle}{\langle u, P_k^2\rangle} \\ &= \frac{\langle u, f_{m-1} P_n P_{k+1}\rangle}{\langle u, P_k^2\rangle} + \beta_k \frac{\langle u, f_{m-1} P_n P_k\rangle}{\langle u, P_k^2\rangle} + \gamma_k \frac{\langle u, f_{m-1} P_n P_{k-1}\rangle}{\langle u, P_k^2\rangle} \\ &= \frac{\langle u, P_{k+1}^2\rangle}{\langle u, P_k^2\rangle} \alpha_{k+1}^{n,m-1} + \beta_k \alpha_k^{n,m-1} + \gamma_k \frac{\langle u, P_{k-1}^2\rangle}{\langle u, P_k^2\rangle} \alpha_{k-1}^{n,m-1}. \end{aligned}$$

Now, using again the three-term recurrence formula (2.1), we get $\gamma_k = \langle u, P_k^2\rangle / \langle u, P_{k-1}^2\rangle$ for $k = 1, 2 \ldots$. Thus,

$$\alpha_k^{n,m} = \gamma_{k+1} \alpha_{k+1}^{n,m-1} + \beta_k \alpha_k^{n,m-1} + \alpha_{k-1}^{n,m-1} \tag{2.6}$$

and $\alpha_{k-1}^{k,1} = \gamma_k$, $\alpha_k^{k,1} = \beta_k$, $\alpha_{k+1}^{k,1} = 1$, for $k = 1, 2, \ldots$.

3. Polynomial Matrices

Let $A(s) := s^p A_p + s^{p-1} A_{p-1} + \cdots + s A_1 + A_0$ be a matrix whose entries are polynomials of degree at most p. The coefficients $(A_k)_{k=0}^p$ are matrices in $\mathbb{C}^{n \times n}$.

Definition 3.1. $\alpha \in \mathbb{C}$ is said to be a zero of $A(s)$ if $\det A(\alpha) = 0$.

Thus it is an important question to find $\det A(s)$ using a finite number of linear steps as an alternative to the natural approach based in terms of determinants of polynomial matrices of dimension $n-1$. On the other hand, the adjoint of the matrix $A(s)$ plays an important role in the transfer function of a linear continuous system. Indeed

$$[A(s)]^{-1} = \frac{1}{\det A(s)} \text{Adj } A(s).$$

Notice that $\deg \det A(s) \leqslant pn$ and thus the complexity of our problem increases with n.

In the last years an increasing attention was paid to some cases of polynomial matrices.

Definition 3.2. Let μ be a $N \times N$ positive definite matrix of measures supported on the real line, and $(P_n)_{n=0}^{+\infty}$ a sequence of matrix polynomials with $\deg P_n(s) = n$. $(P_n)_{n=0}^{+\infty}$ is orthogonal with respect to μ if

$$\int_{\mathbb{R}} P_n(s) d\mu(s) P_m^*(s) = 0, \text{ if } n \neq m, \; n, m \geqslant 0, \tag{3.1}$$

and $(P_n)_{n=0}^{+\infty}$ is orthonormal with respect to μ if

$$\int_{\mathbb{R}} P_n(s) d\mu(s) P_m^*(s) = \delta_{n,m} I_N, \quad n, m \geqslant 0, \tag{3.2}$$

For a positive definite matrix of measures non-degenerate μ having moments of any order, there exists a sequence of orthonormal matrix polynomials with respect to μ. This sequence satisfies a three-term recurrence relation of the form

$$sP_n(s) = A_{n+1} P_{n+1}(s) + B_n P_n(s) + A_n^* P_{n-1}(s) \tag{3.3}$$

with the initial conditions $P_{-1}(s) = 0$ and $P_0(s) = I_N$, where A_n is nonsingular and B_n is hermitian, $n \geqslant 0$.

This three-term recurrence relation characterizes sequence of orthonormal matrix polynomials. In other words, if a family of matrix polynomials satisfies a three-term recurrence relation as (3.3), then this family is orthonormal with respect to some positive definite matrix of measures. This result is known as Favard's Theorem.

Definition 3.3. Let $(P_n)_{n=0}^{+\infty}$ be a family of orthogonal matrix polynomials with respect to μ, the matrix polynomial sequence of the second kind is the family $(Q_n)_{n=0}^{+\infty}$, defined by

$$Q_n(t) = \int \frac{P_n(t) - P_n(x)}{t - x}\, d\mu(x), \quad n \geqslant 0. \tag{3.4}$$

The sequence $(Q_n)_{n=0}^{+\infty}$ satisfies the three-term recurrence relation (3.3), with initial conditions $Q_0(t) = 0$ and $Q_1(t) = A_1^{-1}$.

We get from the three-term recurrence relation that

$$s \begin{bmatrix} P_0(s) \\ P_1(s) \\ P_2(s) \\ \vdots \end{bmatrix} = \begin{bmatrix} B_0 & A_1 & & & \\ A_1^* & B_1 & A_2 & & \\ & A_2^* & B_2 & A_3^* & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} P_0(s) \\ P_1(s) \\ P_2(s) \\ \vdots \end{bmatrix}. \tag{3.5}$$

The infinite dimensional matrix

$$J = \begin{bmatrix} B_0 & A_1 & & & \\ A_1^* & B_1 & A_2 & & \\ & A_2^* & B_2 & A_3^* & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

is said to be the N-Jacobi matrix associated with the family $(P_n)_{n=0}^{+\infty}$. The N-Jacobi matrix plays an important role in the study of the zeros and quadrature formula for the matrix polynomials $(P_n)_{n=0}^{+\infty}$.

Lemma 3.1. ([3]) *For $n \in \mathbb{N}$, the zeros of the matrix polynomials $P_n(s)$ are the same as those of the polynomial* $\det(sI_{nN} - J_{nN})$ *with the same multiplicity, where I_{nN} is the identity matrix of dimension nN and J_{nN} is the N-Jacobi matrix truncated of size nN:*

$$J = \begin{bmatrix} B_0 & A_1 & & & \\ A_1^* & B_1 & A_2 & & \\ & \ddots & \ddots & \ddots & \\ & & A_{n-2}^* & B_{n-2} & A_{n-1} \\ & & & A_{n-1}^* & B_{n-1} \end{bmatrix}. \tag{3.6}$$

The next result is consequence of the previous lemma.

Theorem 3.1. ([3]) 1° *The zeros of* P_n *have a multiplicity less than or equal to* N. *Furthermore* P_n *has* nN *real zeros* (*taking into account their multiplicities*), $n \in \mathbb{N}$.

2° *If* a *is a zero of* P_n *with multiplicity* p, *then* $\operatorname{rank}(P_n(a)) = N - p$.

3° *If we write* $x_{n,k}$ $(k = 1, \ldots, nN)$, *the zeros of* P_n *in increasing order* (*and taking into account their multiplicities*), *then*

$$x_{n+1,k} \leqslant x_{n,k} \leqslant x_{n+1,k+N} \quad \text{for } k = 1, \ldots, nN.$$

4° *If* a *is both a zero of* P_n *and* P_{n+1}, *then* $P_n(a)$ *and* $P_{n+1}(a)$ *do not have a common eigenvector associated to zero.*

5° *If* $x_{n,k}$ *is a zero of* P_n *with multiplicity* N, *then* $P_n(x_{n,k}) = 0$. *Furthermore every complex value of* $x_{n,k}^{1/N}$ *is a zero of the* N *consecutive scalar polynomials* $p_{nN+j}(t) = \det(P_{nN+j}(t))$ $j = 0, 1, \ldots, N-1$. *In this case the real number* $x_{n,k}$ *can not be a zero of the matrix polynomial* P_{n+1}.

6° *Quadrature formula. Associated with every zero* $x_{n,k}$ *of the matrix polynomial* P_n $(k = 1, \ldots, nN)$ *there exists a* $N \times N$ *positive semidefinite matrix* $B_{n,k}$ *of rank one such that*

$$\langle P, Q \rangle_{(P_n)} = \sum_{k=1}^{nN} P(x_{n,k}) B_{n,k} Q^*(x_{n,k}),$$

for P, Q *matrix polynomials satisfying* $\deg(P) + \deg(Q) \leqslant 2n - 1$, *where* $\langle \cdot, \cdot \rangle_{(P_n)}$ *denotes the matrix inner product associated with* $(P_n)_{n=0}^{+\infty}$.

Let A be a $N \times N$ nonsingular matrix, and B be a $N \times N$ Hermitian matrix. We define a sequence of orthonormal matrix polynomials $(U_n^{A,B})_{n=0}^{+\infty}$, by the three-term recurrence relation

$$tU_n^{A,B}(t) = A^* U_{n+1}^{A,B}(t) + BU_n^{A,B}(t) + AU_{n-1}^{A,B}(t), \quad n \geqslant 0, \tag{3.7}$$

with the initial conditions $U_{-1}^{A,B}(t) = 0$ and $U_0^{A,B}(t) = I_N$. The elements of this family are the Chebyshev matrix polynomials of the second kind.

If A is positive definite, then we can deduce an explicit form for the corresponding matrix of measures. Indeed, we introduce the matrix

$$K_{A,B}(z) = \frac{1}{2} A^{-1/2} (B - zI_N) A^{-1/2},$$

for $z = x \in \mathbb{R}$. This matrix is hermitian and we can express $K_{A,B}(x) = U(x)D(x)U^*(x)$, where $D(x)$ is a diagonal matrix, with $d_{i,i}(x) \in \mathbb{R}$, $i = 1, \ldots, N$, and $U(x)^*U(x) = I_N$. Then

$$d\mu_{A,B}(x) = \frac{1}{\pi} A^{-1/2} U(x) S(x) U^*(x) A^{-1/2}\, dx, \tag{3.8}$$

where $S(x) = \operatorname{diag}\,(s_1(x), \ldots, s_N(x))$ and

$$s_i(x) = \begin{cases} \sqrt{1 - d_{i,i}^2(x)} & \text{if} \quad d_{i,i}(x) \in (-1,1), \\ 0 & \text{if} \quad d_{i,i}(x) \notin (-1,1), \end{cases}$$

for $i = 1, \ldots, N$. The support of $\mu_{A,B}$ is

$$\operatorname{supp}\,(\mu_{A,B}) = \left\{ x \in \mathbb{R} :\ \sigma\left(A^{-1/2}(xI_N - B)A^{-1/2} \right) \cap [-2,2] \neq \varnothing \right\}.$$

Thus, we can express the matrix polynomial $U_n^{A,B}(t)$ as

$$U_n^{A,B}(t) = A^{-1/2} U_n \left(\frac{1}{2} A^{-1/2}(B - tI_N)A^{-1/2} \right) A^{-1/2}. \tag{3.9}$$

where $(U_n)_{n=0}^{+\infty}$ is the family of scalar Chebyshev polynomials of the second kind.

4. The Algorithm

For a matrix $A \in \mathbb{C}^{n \times n}$, we presented in [6] an alternative version of Leverrier algorithm for simultaneous computation of the characteristic polynomial $p_A(\lambda)$ of A and the matrix $\operatorname{Adj}\,(\lambda I_n - A)$ in terms of a family of classical orthogonal polynomials (P_k), i. e.

$$p_A(\lambda) = P_n(\lambda) + \sum_{k=0}^{n-1} \hat{a}_{n-k} P_k(\lambda), \tag{4.1}$$

$$\tilde{A}(\lambda) = P_{n-1}(\lambda) I_n + \sum_{k=0}^{n-2} P_k(\lambda) \hat{B}_{n-k-1}. \tag{4.2}$$

The algorithm presented in [6] is a follows:

DATA: $\{\beta_k\}_{k=0}^{n-1},\ \{\gamma_k\}_{k=1}^{n},\ \{r_k\}_{k=0}^{n-1},\ \{s_k\}_{k=1}^{n}$.

Initial Condition: $\hat{B}_{-1} = 0,\ \hat{B}_0 = I_n$.

FOR $k = 1, 2, \ldots, n-1$

$$(4.3)\quad \hat{a}_k = \frac{1}{k}\left[(\beta_{n-k} - r_{n-k})\mathrm{tr}\hat{B}_{k-1} + (\gamma_{n-k+1} - s_{n-k+1})\mathrm{tr}\hat{B}_{k-2} - \mathrm{tr}\left(A\hat{B}_{k-1}\right)\right],$$

$$(4.4)\quad \hat{B}_k = A\hat{B}_{k-1} + \hat{a}_k I - \gamma_{n-k+1}\hat{B}_{k-2} - \beta_{n-k}\hat{B}_{k-1}.$$

END (FOR)

$$\hat{a}_n = \frac{1}{n}\left[(\beta_0 - r_0)\mathrm{tr}\hat{B}_{n-1} + (\gamma_1 - s_1)\mathrm{tr}\hat{B}_{n-2} - \mathrm{tr}\left(A\hat{B}_{n-1}\right)\right].$$

Now, we are interesting in the computation of $a(s) := \det(s^p A_p + \cdots + sA_1 + A_0)$ and $B(s) := \mathrm{Adj}\,(s^p A_p + \cdots + sA_1 + A_0)$. Proceeding in the same way as [7], that is, in the previous algorithm we replace A by $A(s) := -(s^p A_p + \cdots + sA_1 + A_0)$, then we get

$$(4.5)\qquad \tilde{a}(\lambda, s) := \det\,(\lambda I_n - A(s)) = P_n(\lambda) + \sum_{k=0}^{n-1} \hat{a}_{n-k}(s) P_k(\lambda),$$

as well as

$$(4.6)\quad \tilde{B}(\lambda, s) := \mathrm{Adj}\,(\lambda I_n - A(s)) = P_{n-1}(\lambda) I_n + \sum_{k=0}^{n-2} P_k(\lambda)\hat{B}_{n-k-1}(s).$$

Thus, from (4.3) and (4.4) we get

$$(4.7)\qquad \begin{aligned} k\hat{a}_k(s) &= (\beta_{n-k} - r_{n-k})\mathrm{tr}\hat{B}_{k-1}(s) - \mathrm{tr}\left(A(s)\hat{B}_{k-1}(s)\right) \\ &\quad + (\gamma_{n-k+1} - s_{n-k+1})\mathrm{tr}\hat{B}_{k-2}(s), \quad k = 1, \ldots, n, \end{aligned}$$

as well as

$$(4.8)\quad \hat{B}_k(s) = \hat{a}_k(s) I_n - \gamma_{n-k+1}\hat{B}_{k-2}(s) - \beta_{n-k}\hat{B}_{k-1}(s) + A(s)\hat{B}_{k-1}(s),$$

for $k = 1, \ldots, n-1$.

Thus, if $\lambda = 0$ in (4.5) and (4.6) then we get

$$(4.9)\qquad a(s) = \tilde{a}(0, s) = P_n(0) + \sum_{k=0}^{n-1} \hat{a}_{n-k}(s) P_k(0),$$

$$(4.10)\qquad B(s) = \tilde{B}(0, s) = P_{n-1}(0) I_n + \sum_{k=0}^{n-2} P_k(0)\hat{B}_{n-k-1}(s).$$

Taking into account $\deg(P_k(s)) = k$ for all $k \geqslant 0$, from (4.7) and (4.8) we can assure that the degrees of the polynomial $\hat{a}_k(s),\ k = 1, 2, \ldots, n$ and the polynomial matrix $\hat{B}_k(s),\ k = 1, 2, \ldots, n-1$ are at most equal to kp. Thus for $\hat{a}_k(s)$ and $\hat{B}_k(s)$ we get the expansions

$$\begin{aligned} \hat{a}_k(s) &= \sum_{j=0}^{kp} a_{k,j} P_j(s), \ \ a_{k,j} \in \mathbb{C}, \\ \hat{B}_k(s) &= \sum_{j=0}^{kp} P_j(s) B_{k,j}, \ \ B_{k,j} \in \mathbb{C}^{n\times n}. \end{aligned} \tag{4.11}$$

Substituting (4.11) in (4.7), we get

$$\begin{aligned} k\sum_{j=0}^{kp} a_{k,j}P_j(s) &= \operatorname{tr}\Bigg((\beta_{n-k} - r_{n-k}) \sum_{j=0}^{(k-1)p} P_j(s) B_{k-1,j} \\ &\qquad + (\gamma_{n-k+1} - s_{n-k+1}) \sum_{j=0}^{(k-2)p} P_j(s) B_{k-2,j} \\ &\qquad + (s^p A_p + \cdots + sA_1 + A_0) \sum_{j=0}^{(k-1)p} P_j(s) B_{k-1,j} \Bigg) \\ &= \operatorname{tr}\Bigg((\beta_{n-k} - r_{n-k}) \sum_{j=0}^{(k-1)p} P_j(s) B_{k-1,j} \\ &\qquad + (\gamma_{n-k+1} - s_{n-k+1}) \sum_{j=0}^{(k-2)p} P_j(s) B_{k-2,j} \\ &\qquad + \sum_{m=0}^{p} \sum_{j=0}^{(k-1)p} s^m P_j(s) A_m B_{k-1,j} \Bigg). \end{aligned}$$

If we replace (2.5) in the previous expression, we get

$$\begin{aligned} k\sum_{j=0}^{kp} a_{k,j}P_j(s) &= \sum_{j=p}^{kp} \alpha_j^{j-p,p} \operatorname{tr}\left(A_p B_{k-1,j-p}\right) P_j(s) \\ &\quad + \sum_{j=p-1}^{kp-1} \left(\sum_{i=0}^{1} \alpha_j^{j-p+1,p-i} \operatorname{tr}\left(A_{p-i} B_{k-1,j-p+1}\right) \right) P_j(s) \\ &\quad \vdots \\ &\quad + \sum_{j=1}^{(k-1)p+1} \left(\sum_{i=0}^{p-1} \alpha_j^{j-1,p-i} \operatorname{tr}\left(A_{p-i} B_{k-1,j-1}\right) \right) P_j(s) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=0}^{(k-1)p}\left(\sum_{i=0}^{p}\alpha_j^{j,p-i}\mathrm{tr}\,(A_{p-i}B_{k-1,j})+(\beta_{n-k}-r_{n-k})\mathrm{tr}B_{k-1,j}\right)P_j(s)\\
&+\sum_{j=0}^{(k-1)p-1}\left(\sum_{i=0}^{p-1}\alpha_j^{j+1,p-i}\mathrm{tr}\,(A_{p-i}B_{k-1,j+1})\right)P_j(s)\\
&\vdots\\
&+\sum_{j=0}^{(k-2)p+1}\left(\sum_{i=0}^{1}\alpha_j^{j+p-1,p-i}\mathrm{tr}\,(A_{p-i}B_{k-1,j+p-1})\right)P_j(s)\\
&+\sum_{j=0}^{(k-2)p}\left(\alpha_j^{j+p,p}\mathrm{tr}\,(A_pB_{k-1,j+p})+(\gamma_{n-k+1}-s_{n-k+1})\mathrm{tr}B_{k-2,j}\right)P_j(s).
\end{aligned}
$$

Identifying coefficients we get

$$
\begin{aligned}
(4.12)\ ka_{k,0} &= \delta_k\mathrm{tr}B_{k-1,0}+\eta_k\mathrm{tr}B_{k-2,0}+\sum_{i=0}^{p}\alpha_0^{0,p-i}\mathrm{tr}\,(A_{p-i}B_{k-1,0})\\
&\quad+\sum_{l=0}^{p-1}\sum_{i=0}^{l}\alpha_0^{p-l,p-i}\mathrm{tr}\,(A_{p-i}B_{k-1,p-l})\,,\\
ka_{k,1} &= \delta_k\mathrm{tr}B_{k-1,1}+\eta_k\mathrm{tr}B_{k-2,1}+\sum_{l=p-1}^{p}\sum_{i=0}^{l}\alpha_1^{l-p+1,p-i}\mathrm{tr}\,(A_{p-i}B_{k-1,l-p+1})\\
&\quad+\sum_{l=0}^{p-1}\sum_{i=0}^{l}\alpha_1^{p-l+1,p-i}\mathrm{tr}\,(A_{p-i}B_{k-1,p-l+1})\,,\\
&\ \vdots\\
ka_{k,p-1} &= \delta_k\mathrm{tr}B_{k-1,p-1}+\eta_k\mathrm{tr}B_{k-2,p-1}+\sum_{l=1}^{p}\sum_{i=0}^{l}\alpha_{p-1}^{l-1,p-i}\mathrm{tr}\,(A_{p-i}B_{k-1,l-1})\\
&\quad+\sum_{l=0}^{p-1}\sum_{i=0}^{l}\alpha_{p-1}^{2p-l-1,p-i}\mathrm{tr}\,(A_{p-i}B_{k-1,2p-l+1})\,,\\
ka_{k,j} &= \delta_k\mathrm{tr}B_{k-1,j}+\eta_k\mathrm{tr}B_{k-2,j}+\sum_{l=0}^{p}\sum_{i=0}^{l}\alpha_j^{j-p+l,p-i}\mathrm{tr}\,(A_{p-i}B_{k-1,j-p+l})\\
&\quad+\sum_{l=0}^{p-1}\sum_{i=0}^{l}\alpha_{p-1}^{j+p-l,p-i}\mathrm{tr}\,(A_{p-i}B_{k-1,j+p-l})\,,
\end{aligned}
$$

for $j=p,\ldots,(k-2)p$, and

$$\begin{aligned}
ka_{k,(k-2)p+1} &= \delta_k \mathrm{tr} B_{k-1,(k-2)p+1} \\
&\quad + \sum_{l=0}^{p} \sum_{i=0}^{l} \alpha_{(k-2)p+1}^{(k-3)p+l+1,p-i} \mathrm{tr}\left(A_{p-i} B_{k-1,(k-3)p+l+1}\right) \\
&\quad + \sum_{l=1}^{p-1} \sum_{i=0}^{l} \alpha_{(k-2)p+1}^{(k-1)p-l+1,p-i} \mathrm{tr}\left(A_{p-i} B_{k-1,(k-1)p-l+1}\right), \\
&\ \vdots \\
ka_{k,(k-1)p-1} &= \delta_k \mathrm{tr} B_{k-1,(k-1)p-1} \\
&\quad + \sum_{l=0}^{p} \sum_{i=0}^{l} \alpha_{(k-1)p-1}^{(k-2)p+l-1,p-i} \mathrm{tr}\left(A_{p-i} B_{k-1,(k-2)p+l-1}\right) \\
&\quad + \sum_{i=0}^{p-1} \alpha_{(k-1)p-1}^{(k-1)p,p-i} \mathrm{tr}\left(A_{p-i} B_{k-1,(k-1)p}\right), \\
ka_{k,(k-1)p} &= \delta_k \mathrm{tr} B_{k-1,(k-1)p} + \sum_{l=0}^{p} \sum_{i=0}^{l} \alpha_{(k-1)p}^{(k-2)p+l,p-i} \mathrm{tr}\left(A_{p-i} B_{k-1,(k-2)p+l}\right), \\
ka_{k,(k-1)p+1} &= \sum_{l=0}^{p} \sum_{i=0}^{l} \alpha_{(k-1)p+1}^{(k-2)p+l+1,p-i} \mathrm{tr}\left(A_{p-i} B_{k-1,(k-2)p+l+1}\right), \\
&\ \vdots \\
ka_{k,kp-1} &= \sum_{l=0}^{1} \sum_{i=0}^{l} \alpha_{kp-1}^{(k-1)p+l-1,p-i} \mathrm{tr}\left(A_{p-i} B_{k-1,(k-1)p+l-1}\right), \\
ka_{k,kp} &= \alpha_{kp}^{(k-1)p,p} \mathrm{tr}\left(A_p B_{k-1,(k-1)p}\right),
\end{aligned}$$

where $\delta_k = \beta_{n-k} - r_{n-k}$, $\eta_k = \gamma_{n-k+1} - s_{n-k+1}$.

In an analogous way, substituting (4.11) in (4.8), we deduce

$$\begin{aligned}
\sum_{j=0}^{kp} P_j(s) B_{k,j} &= \sum_{j=0}^{kp} a_{k,j} P_j(s) I_n - \sum_{m=0}^{p} \sum_{j=0}^{(k-1)p} s^m P_j(s) A_m B_{k-1,j} \\
&\quad - \sum_{j=0}^{(k-1)p} \beta_{n-k} P_j(s) B_{k-1,j} - \sum_{j=0}^{(k-2)p} \gamma_{n-k+1} P_j(s) B_{k-2,j}.
\end{aligned}$$

Replacing (2.5) in the previous expression, we get

$$\sum_{j=0}^{kp} P_j(s) B_{k,j} = \sum_{j=0}^{kp} a_{k,j} P_j(s) I_n - \sum_{j=p}^{kp} P_j(s) \alpha_j^{j-p,p} A_p B_{k-1,j-p}$$

$$
\begin{aligned}
&- \sum_{j=p-1}^{kp-1} P_j(s) \left(\sum_{i=0}^{1} \alpha_j^{j-p+1,p-i} A_{p-i} \right) B_{k-1,j-p+1} \\
&\vdots \\
&- \sum_{j=1}^{(k-1)p+1} P_j(s) \left(\sum_{i=0}^{p-1} \alpha_j^{j-1,p-i} A_{p-i} \right) B_{k-1,j-1} \\
&- \sum_{j=0}^{(k-1)p} P_j(s) \left(\beta_{n-k} I_n + \sum_{i=0}^{p} \alpha_j^{j,p-i} A_{p-i} \right) B_{k-1,j} \\
&- \sum_{j=0}^{(k-1)p-1} P_j(s) \left(\sum_{i=0}^{p-1} \alpha_j^{j+1,p-i} A_{p-i} \right) B_{k-1,j+1} \\
&\vdots \\
&- \sum_{j=0}^{(k-2)p+1} P_j(s) \left(\sum_{i=0}^{1} \alpha_j^{j+p-1,p-i} A_{p-i} \right) B_{k-1,j+p-1} \\
&- \sum_{j=0}^{(k-2)p} P_j(s) \left(\alpha_j^{j+p,p} A_p B_{k-1,j+p} + \gamma_{n-k+1} B_{k-2,j} \right).
\end{aligned}
$$

Identifying coefficients we obtain

$$
\begin{aligned}
B_{k,0} &= a_{k,0} I_n - \beta_{n-k} B_{k-1,0} - \gamma_{n-k+1} B_{k-2,0} \\
&\quad - \sum_{l=0}^{p-1} \sum_{i=0}^{l} \alpha_0^{p-l,p-i} A_{p-i} B_{k-1,p-l} - \sum_{i=0}^{p} \alpha_0^{0,p-i} A_{p-i} B_{k-1,0}, \\
B_{k,1} &= a_{k,1} I_n - \beta_{n-k} B_{k-1,1} - \gamma_{n-k+1} B_{k-2,1} \\
&\quad - \sum_{l=p-1}^{p} \sum_{i=0}^{l} \alpha_1^{l-p+1,p-i} A_{p-i} B_{k-1,l-p+1} \\
&\quad - \sum_{l=0}^{p-1} \sum_{i=0}^{l} \alpha_1^{p-l+1,p-i} A_{p-i} B_{k-1,p-l+1}, \\
&\vdots \\
B_{k,p-1} &= a_{k,p-1} I_n - \beta_{n-k} B_{k-1,p-1} - \gamma_{n-k+1} B_{k-2,p-1} \\
&\quad - \sum_{l=1}^{p} \sum_{i=0}^{l} \alpha_{p-1}^{l-1,p-i} A_{p-i} B_{k-1,l-1} \\
&\quad - \sum_{l=0}^{p-1} \sum_{i=0}^{l} \alpha_{p-1}^{2p-l-1,p-i} A_{p-i} B_{k-1,2p-l+1}, \\
B_{k,j} &= a_{k,j} I_n - \beta_{n-k} B_{k-1,j} - \gamma_{n-k+1} B_{k-2,j}
\end{aligned}
\tag{4.13}
$$

$$-\sum_{l=0}^{p}\sum_{i=0}^{l}\alpha_{j}^{j-p+l,p-i}A_{p-i}B_{k-1,j-p+l}$$
$$-\sum_{l=0}^{p-1}\sum_{i=0}^{l}\alpha_{p-1}^{j+p-l,p-i}A_{p-i}B_{k-1,j+p-l},$$

for $j = p,\ldots,(k-2)p$, and

$$\begin{aligned}
B_{k,(k-2)p+1} &= a_{k,(k-2)p+1}I_n - \beta_{n-k}B_{k-1,(k-2)p+1}\\
&\quad -\sum_{l=0}^{p}\sum_{i=0}^{l}\alpha_{(k-2)p+1}^{(k-3)p+l+1,p-i}A_{p-i}B_{k-1,(k-3)p+l+1}\\
&\quad -\sum_{l=1}^{p-1}\sum_{i=0}^{l}\alpha_{(k-2)p+1}^{(k-1)p-l+1,p-i}A_{p-i}B_{k-1,(k-1)p-l+1},\\
&\;\vdots\\
B_{k,(k-1)p-1} &= a_{k,(k-1)p-1}I_n - \beta_{n-k}B_{k-1,(k-1)p-1}\\
&\quad -\sum_{l=0}^{p}\sum_{i=0}^{l}\alpha_{(k-1)p-1}^{(k-2)p+l-1,p-i}A_{p-i}B_{k-1,(k-2)p+l-1}\\
&\quad -\sum_{i=0}^{p-1}\alpha_{(k-1)p-1}^{(k-1)p,p-i}A_{p-i}B_{k-1,(k-1)p},\\
B_{k,(k-1)p} &= a_{k,(k-1)p}I_n - \beta_{n-k}B_{k-1,(k-1)p}\\
&\quad -\sum_{l=0}^{p}\sum_{i=0}^{l}\alpha_{(k-1)p}^{(k-2)p+l,p-i}A_{p-i}B_{k-1,(k-2)p+l},\\
B_{k,(k-1)p+1} &= a_{k,(k-1)p+1}I_n\\
&\quad -\sum_{l=0}^{p}\sum_{i=0}^{l}\alpha_{(k-1)p+1}^{(k-2)p+l+1,p-i}A_{p-i}B_{k-1,(k-2)p+l+1},\\
&\;\vdots\\
B_{k,kp-1} &= a_{k,kp-1}I_n - \sum_{l=0}^{1}\sum_{i=0}^{l}\alpha_{kp-1}^{(k-1)p+l-1,p-i}A_{p-i}B_{k-1,(k-1)p+l-1},\\
B_{k,kp} &= a_{k,kp}I_n - \alpha_{kp}^{(k-1)p,p}A_pB_{k-1,(k-1)p}.
\end{aligned}$$

Thus, the algorithm can be described as follows

DATA: $\{\beta_k\}_{k=0}^{n-1},\ \{\gamma_k\}_{k=1}^{n},\ \{r_k\}_{k=0}^{n-1},\ \{s_k\}_{k=1}^{n}.$

Initial Condition: $\hat{a}_0(s) = 1,\ \hat{B}_{-1}(s) = 0,\ \hat{B}_0(s) = I_n$, and $B_{i,j} = 0$ if $i,j < 0$ or $i < pj$.

FOR $k = 1, \dots, n-1$

- From $\hat{B}_{k-2}(s)$ and $\hat{B}_{k-1}(s)$ and taking into account (4.12) we get $\hat{a}_k(s)$.
- From (4.13) we get $\hat{B}_k(s)$.

END (FOR)

From $\hat{B}_{n-2}(s)$ and $\hat{B}_{n-1}(s)$ taking into account (4.12) we get $\hat{a}_n(s)$.

5. Examples

Example 5.1. Consider the polynomial matrix

$$A(s) = \begin{bmatrix} s^2 - s & -s^2 + s - 1 & 0 \\ 0 & 3 & s^2 + 2s \\ s^2 - 1 & -s^2 & s^2 + s \end{bmatrix},$$

so that $p = 2$, $n = 3$ and

$$A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 3 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

It is easy to verify that $\det A(s) = 3s^4 - s^3 - 4s^2 + 2s$,

$$\text{Adj } A(s) = \begin{bmatrix} s^4 + 2s^3 + 3s^2 + 3s & s^4 + s & -s^4 - s^3 + s^2 - 2s \\ s^4 + 2s^3 - s^2 - 2s & s^4 - s^2 & -s^4 - s^3 + 2s^2 \\ s^2 - 1 & -s^2 & s^2 + s \end{bmatrix}.$$

We apply the above algorithm using the Chebyshev basis (Chebyshev polynomials of first kind), $(T_n)_{n=0}^{+\infty}$. The corresponding data are $\beta_n = r_n = 0$, $n \geqslant 0$ and $\gamma_1 = 1/2$, $s_1 = 0$, $\gamma_n = 1/4$, $s_n = -1/4$, $n \geqslant 2$.

From (4.12) and (4.13) we get the following results.

For $\mathbf{k = 1}$:

$$a_{1,0} = 4, \quad a_{1,1} = 0, \quad a_{1,2} = 2;$$

$$B_{1,0} = \frac{1}{2} \begin{bmatrix} 7 & 3 & 0 \\ 0 & 2 & -1 \\ 1 & 1 & 7 \end{bmatrix}, \quad B_{1,2} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{bmatrix}, \quad B_{1,2} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ -1 & 1 & 1 \end{bmatrix}.$$

For $\mathbf{k = 2}$:

$$a_{2,0} = 4, \quad a_{2,1} = \frac{3}{2}, \quad a_{2,2} = 7, \quad a_{2,3} = 2, \quad a_{2,4} = 2;$$

$$B_{2,0} = \frac{1}{8}\begin{bmatrix} 19 & 3 & 1 \\ -1 & 3 & 5 \\ 12 & 8 & 16 \end{bmatrix}, \quad B_{2,1} = \frac{1}{4}\begin{bmatrix} 18 & 4 & -11 \\ -2 & 0 & -3 \\ 0 & -4 & -12 \end{bmatrix},$$

$$B_{2,2} = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & 0 & 3 \end{bmatrix}, \quad B_{2,3} = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_{2,4} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

For $\mathbf{k = 3}$:

$$a_{3,0} = \frac{9}{8}, \quad a_{3,1} = \frac{5}{4}, \quad a_{3,2} = 0, \quad a_{3,3} = -1, \quad a_{3,4} = 3, \quad a_{3,5} = 0 = a_{3,6}.$$

Hence, from (4.9) we get

$$\det A(s) = 3T_4(s) - T_3(s) - T_2(s) + \frac{5}{4}T_1(s) - \frac{7}{8}T_0(s),$$

and, from (4.10) we obtain

$$\begin{aligned} \operatorname{Adj} A(s) \;=\; & T_4(s)\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} + T_3(s)\begin{bmatrix} 2 & 0 & -1 \\ 2 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} + T_2(s)\begin{bmatrix} 4 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & 0 & 3 \end{bmatrix} \\ & + T_1(s)\frac{1}{4}\begin{bmatrix} 18 & 4 & -11 \\ -2 & 0 & -3 \\ 0 & -4 & -12 \end{bmatrix} + T_0(s)\frac{1}{8}\begin{bmatrix} 15 & 3 & 1 \\ -1 & -1 & 5 \\ 12 & 8 & 12 \end{bmatrix}. \end{aligned}$$

Example 5.2. Consider the matrices

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Taking into account the three-term recurrence relation (3.7), we get

$$U_4^{A,B}(t) = \begin{bmatrix} 13t^4 - 32t^3 + 24t^2 - 6t & 21t^4 - 52t^3 + 39t^2 - 8t \\ 21t^4 - 52t^3 + 39t^2 - 8t & 34t^4 - 84t^3 + 63t^2 - 14t \end{bmatrix}. \tag{5.1}$$

We need to compute $\det(U_4^{A,B}(t))$ to find the zeros of $U_4^{A,B}(t)$. For the matrix polynomial $U_4^{A,B}(t)$, $p = 4$, $n = 2$ and

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; A_1 = -\begin{bmatrix} 6 & 8 \\ 8 & 14 \end{bmatrix}, \; A_2 = \begin{bmatrix} 24 & 39 \\ 39 & 63 \end{bmatrix},$$

$$A_3 = -\begin{bmatrix} 32 & 52 \\ 52 & 84 \end{bmatrix}, \; A_4 = \begin{bmatrix} 13 & 21 \\ 21 & 34 \end{bmatrix}.$$

We apply the algorithm of the previous section, using the Chebyshev basis (Chebyshev polynomials of second kind), $(U_n)_{n=0}^{+\infty}$. The corresponding data are $\beta_n = r_n = 0$, $n \geqslant 0$ and $\gamma_n = 1/4$, $n \geqslant 1$ and $s_1 = 0$, $s_n = -1/4$, $n \geqslant 2$.

From (4.12) and (4.13) we get:

For $\mathbf{k} = \mathbf{1}$:

$$a_{1,0} = \frac{221}{8}, \quad a_{1,1} = -78, \quad a_{1,2} = \frac{489}{4}, \quad a_{1,3} = -116, \quad a_{1,4} = 47,$$

and

$$B_{1,0} = \frac{1}{8}\begin{bmatrix} 160 & -99 \\ -99 & 61 \end{bmatrix}, \ B_{1,1} = \begin{bmatrix} -56 & 34 \\ 34 & -22 \end{bmatrix}, \ B_{1,2} = \frac{1}{4}\begin{bmatrix} 354 & -219 \\ -219 & 135 \end{bmatrix},$$
$$B_{1,3} = \begin{bmatrix} -84 & 52 \\ 52 & -32 \end{bmatrix}, \ B_{1,4} = \begin{bmatrix} 34 & -21 \\ -21 & 13 \end{bmatrix}.$$

For $\mathbf{k} = \mathbf{2}$:

$$a_{2,0} = \frac{2265}{128}, \quad a_{2,1} = -\frac{209}{4}, \quad a_{2,2} = 93, \quad a_{2,3} = -\frac{225}{2}, \quad a_{2,4} = \frac{177}{2},$$
$$a_{2,5} = -20, \quad a_{2,6} = -\frac{69}{4}, \quad a_{2,7} = 4, \quad a_{2,8} = 1.$$

Thus,

$$\hat{a}_2(t) = \sum_{j=0}^{8} a_{2,j} U_j(t) = t^8 + 4t^7 - 19t^6 - 26t^5 + 111t^4 - 90t^3 + 20t^2 + \frac{1}{4}.$$

Notice that is not necessary to compute $\hat{a}_1(t)$ because $U_1(0) = 0$. Then from (4.9), the determinant of $U_4^{A,B}(t)$ is

$$\det U_4^{A,B}(t) = U_2(0) + U_0(0)\hat{a}_2(t) = t^8 + 4t^7 - 19t^6 - 26t^5 + 111t^4 - 90t^3 + 20t^2.$$

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